Normal Soft int-Groups

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Abstract

In this paper, we define normal soft int-groups and derive their some basic properties. We also investigate some relations on α-inclusion, soft product and normal soft int-groups. Then we define normalizer, quotient group and give some theorems concerning these concepts.

Keywords: Soft set, soft product, soft int-group, normal soft int-group, quotient group.
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1. Introduction

Zadeh [32] introduced the notion of a fuzzy set in 1965 to deal with problems that contains uncertainties. In 1971, Rosenfeld [30] defined the fuzzy subgroup of a group. Rosenfeld's groups made important contributions to the development of fuzzy abstract algebra. Since then, various works have studied analogues of results derived from classical group theory, such as [1, 3, 4, 7, 8, 9, 12, 13, 21, 22, 23, 27, 28, 29]. All above papers and others are combined by Mordeson et al. [26] in the book titled Fuzzy Group Theory.

In 1999, Molodtsov [25] introduced another theory, called soft sets, in order to deal with uncertainty. Then, Maji et al. [24] defined the operations of soft sets, Çağman and Enginoğlu [11] modified these definition and operations of soft sets and Ali et al. [6] gave some new algebraic operations for soft sets. Sezgin and Atagün [31] analyzed the operations of soft sets. Using these definitions, researches have been very active on the soft set theory and many important results have been obtained in theoretical and practical aspects.

The works of the algebraic structure of soft sets was first started by Aktaş and Çağman [5]. They presented the notion of the soft group and derived its some basic properties. Jun [17] and Jun and Park [18] introduced soft BCK/BCI-algebras and its application in ideal theory. Feng et al. [15] worked on soft semirings, soft subsemirings, soft ideals, idealistic soft semirings and

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soft semiring homomorphisms. Jun et al. [19] introduced notions of soft ordered semigroup, soft ideal, and idealistic soft ordered semigroup with their related properties. Acar et al. [2] gave the notion of soft rings and investigated their properties.

Çağman et al. [10] gave a new kind of definition of soft group (soft int-group) in a soft set depending on inclusion relation and intersection of sets. This definition is completely different then the definition of soft group in [5]. Kaygısız [20] presented some supplementary properties of soft sets and soft int-groups and gave some relations on α-inclusion, soft product and soft int-groups.

In this paper, we present normal soft int-groups and investigate their related properties. We then obtained some relations on α-inclusion, soft product and normal soft int-groups. In addition, we defined normalizer, quotient group and give some theorems concerning these concepts.

2. Preliminaries

2.1. Soft sets

In this section, we present basic definitions of soft set theory according to [11]. For more detail see the papers [24, 25].

Throughout this work, U refers to an initial universe, E is a set of parameters and $P(U)$ is the power set of U. $\subset$ and $\supset$ stands for proper subset and superset, respectively.

Çağman and Enginoğlu [11] modified the definition of soft set defined by Molodtsov [25] as follows;

Definition 2.1. [25] For any subset A of E, a soft set f_A over U is a set, defined by a function f_A, representing a mapping

$$f_A : E \longrightarrow P(U) \text{ such that } f_A(x) = \phi \text{ if } x \notin A.$$

A soft set over U can also be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) : x \in E,\ f_A(x) \in P(U)\}.$$

Note that the set of all soft sets over U will be denoted by $S(U)$. From here on, "soft set" will be used without over U.

Definition 2.2. [11] Let $f_A \in S(U)$. If $f_A(x) = \phi$ for all $x \in E$, then f_A is called an empty soft set and denoted by Φ_A.

If $f_A(x) = U$ for all $x \in A$, then f_A is called A-universal soft set and denoted by $f_{\widetilde{A}}$.

If $f_A(x) = U$, for all $x \in E$, then f_A is called a universal soft set and denoted by $f_{\widetilde{E}}$.

Definition 2.3. [24] If $f_A \in S(U)$, then the image (value class) of f_A is defined by $\mathrm{Im}(f_A) = \{f_A(x) : x \in A\}$ and if $A = E$, then $\mathrm{Im}(f_E)$ is called image of E under f_A.

Definition 2.4. [20] Let $f_A : E \longrightarrow P(U)$ be a soft set and $K \subseteq E$. Then, the image of set K under f_A is defined by $f_A(K) = \bigcup\limits_{x \in K} f_A(x)$.

Note that $f_A(K) = \phi$ if $K = \phi$.

Definition 2.5. [11] Let $f_A, f_B \in S(U)$. Then, f_A is a soft subset of f_B, denoted by $f_A \widetilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.

f_A is called a soft proper subset of f_B, denoted by $f_A \widetilde{\subset} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$ and $f_A(x) \neq f_B(x)$ for at least one $x \in E$.

f_A and f_B are called soft equal, denoted by $f_A = f_B$, if and only if $f_A(x) = f_B(x)$ for all $x \in E$.

Definition 2.6. [11] Let $f_A, f_B \in S(U)$. Then, union $f_A \widetilde{\cup} f_B$ and intersection $f_A \widetilde{\cap} f_B$ of f_A and f_B are defined by

$$\left(f_A \widetilde{\cup} f_B\right)(x) = f_A(x) \cup f_B(x), \quad \left(f_A \widetilde{\cap} f_B\right)(x) = f_A(x) \cap f_B(x)$$

for all $x \in E$, respectively.

2.2. Definitions and basic properties of soft int-groups

In this section, we introduce the concepts of soft int-groups, α-inclusion, soft product and their basic properties according to papers by Çağman et al. [10] and Kaygısız [20].

Note that definitions and propositions are changed according to our notations.

Definition 2.7. [10] Let G be a group and $f_G \in S(U)$. Then, f_G is called a soft intersection groupoid over U if $f_G(xy) \supseteq f_G(x) \cap f_G(y)$ for all $x, y \in G$.

f_G is called a soft intersection group (int-group) over U if the soft intersection groupoid satisfies $f_G(x^{-1}) = f_G(x)$ for all $x \in G$.

Note that the condition $f_G(x^{-1}) = f_G(x)$ is equivalent to $f_G(x^{-1}) \supseteq f_G(x)$ for all $x \in G$.

Throughout this paper, G denotes an arbitrary group with identity element e and the set of all soft int-groups with parameter set G over U will be denoted by $S_G(U)$, unless otherwise stated. For short, instead of "f_G is a soft int-group with the parameter set G over U" we say "f_G is a soft int-group".

Theorem 2.8. *[10] Let $f_G \in S_G(U)$. Then $f_G(e) \supseteq f_G(x)$ for all $x \in G$.*

Definition 2.9. [10] Let G be a group. Then, the soft set f_G is called an Abelian soft set if $f_G(xy) = f_G(yx)$ for all $x,y \in G$.

Definition 2.10. [10] Let $f_G \in S(U)$. Then, e-set of f_G, denoted by e_{f_G}, is defined as

$$e_{f_G} = \{x \in G : f_G(x) = f_G(e)\}.$$

Theorem 2.11. *[10] If $f_G \in S_G(U)$, then $e_{f_G} \leq G$.*

We modified the extension principal defined in [10] as follows.

Definition 2.12. [10]Let φ be a function from A into B and $f_A, f_B \in S(U)$. Then, soft image of f_A under φ and soft pre-image (or soft inverse image) of f_B under φ are the soft sets $\varphi(f_A)$ and $\varphi^{-1}(f_B)$ such that

$$\varphi(f_A)(y) = \begin{cases} \bigcup \{f_A(x) : x \in A,\ \varphi(x) = y\} & \text{if } \varphi^{-1}(y) \neq \phi \\ \phi & \text{otherwise} \end{cases}$$

for all $y \in B$ and $\varphi^{-1}(f_B)(x) = f_B(\varphi(x))$ for all $x \in A$, respectively.

Clearly, $\varphi(f_A) = f_B$ such that $f_B(y) = f_A\left(\varphi^{-1}(y)\right)$ for all $y \in B$ and $\varphi^{-1}(f_B) = f_A$ such that $f_A(x) = f_B(\varphi(x))$ for all $x \in A$.

Theorem 2.13. *[20] If $f_G \in S_G(U)$ and $H \leq G$, then the restriction $f_G|_H \in S_H(U)$.*

Theorem 2.14. *[20] Let $A_i \leq G$ for all $i \in I$ and $\{f_{A_i} : i \in I\}$ be a family of soft int-groups. Then, $\widetilde{\bigcap_{i \in I}} f_{A_i} \in S_G(U)$.*

Definition 2.15. [10] Let $f_A \in S(U)$ and $\alpha \in P(U)$. Then, α-inclusion of the soft set f_A, denoted by f_A^{α}, is defined as

$$f_A^{\alpha} = \{x \in A : f_A(x) \supseteq \alpha\}.$$

If $\alpha = \phi$ then the set $f_A^{*} = \{x \in A : f_A(x) \neq \phi\}$ is called support of f_A.

In particular, the set $f_A^{\alpha'} = \{x \in A : f_A(x) \supset \alpha\}$ is called strong α-inclusion.

Corollary 2.16. *[20] For any soft set f_A, if $\alpha \subseteq \beta$ and $\alpha, \beta \in P(U)$, then $f_A^{\beta} \subseteq f_A^{\alpha}$.*

Definition 2.17. [20] Let $f_A \in S(U)$ and $\alpha \in P(U)$. Then, the soft set $f_{A\alpha}$, defined by, $f_{A\alpha}(x) = \alpha$, for all $x \in A$, is called $A-\alpha$ soft set. If A is a singleton, say $\{w\}$, then $f_{w\alpha}$ is called a soft singleton (or soft point). If $\alpha = U$, then $f_{AU} = f_{\widetilde{A}}$ is the characteristic function of A.

Theorem 2.18. *[20]Let $f_G \in S(U)$ and $\alpha \in P(U)$. Then, $f_G \in S_G(U)$ if and only if f_G^{α} is a subgroup of G, whenever it is nonempty.*

Definition 2.19. [20] Let $f_G \in S_G(U)$. Then, the subgroups f_G^{α} are called soft level subgroups of G for any $\alpha \in P(U)$.

Definition 2.20. [20] Let G be a group and $A, B \subseteq G$. Then, soft product of soft sets f_A and f_B is defined as

$$(f_A * f_B)(x) = \bigcup\{f_A(u) \cap f_B(v) : uv = x,\ u, v \in G\}$$

for all $x \in G$ and inverse of f_A is defined as

$$f_A^{-1}(x) = f_A(x^{-1})$$

for all $x \in G$.

Theorem 2.21. *[20] Let G be a group and $f_{x\alpha}$,$f_{y\beta} \in S_G(U)$. Then, for any x,$y \in G$ and $\phi \subset \alpha$,$\beta \subseteq U$, $f_{x\alpha} * f_{y\beta} = f_{(xy)(\alpha\cap\beta)}$.*

Theorem 2.22. *[20] Let A,B,$C \subseteq G$ and f_A,f_B,$f_C \in S(U)$. Then,*

$$(f_A * f_B) * f_C = f_A * (f_B * f_C)$$

Corollary 2.23. *[20] Let $A \subseteq G$ and f_A,$f_{u\alpha}$ be soft sets where $\alpha = f_A(A)$. Then, for any x,$u \in G$,*

$$(f_{u\alpha} * f_A)(x) = f_A(u^{-1}x) \text{ and } (f_A * f_{u\alpha})(x) = f_A(xu^{-1})$$

Corollary 1. Let $f_A \in S_G(U)$. Then,

$$f_A \widetilde{\subseteq} f_A^{-1} \Leftrightarrow f_A^{-1} \widetilde{\subseteq} f_A \Leftrightarrow f_A = f_A^{-1}.$$

.

Theorem 2.24. *[20] Let $f_G \in S(U)$. Then, f_G is a soft int-group if and only if f_G satisfies the following conditions:*

1. $(f_G * f_G) \widetilde{\subseteq} f_G$,
2. $f_G^{-1} = f_G$ *(or* $f_G \widetilde{\subseteq} f_G^{-1}$ *or* $f_G^{-1} \widetilde{\subseteq} f_G$*).*

Theorem 2.25. *[20] Let f_A,$f_B \in S_G(U)$. Then, $(f_A * f_B)$ is a soft int-group if and only if $f_A * f_B = f_B * f_A$.*

3. Normal soft int-groups and cosets

Normal subgroup is very important concept in classical group theory. In this section, we introduce the notion of normal soft int-groups, and obtain the analogues to the classical group theory and fuzzy group theory.

Definition 3.1. Let $f_A \in S_G(U)$. Then, f_A is called soft normal in G (or normal soft int-group), if f_A is an Abelian soft set.

Let $NS_G(U)$ denotes the set of all normal soft int-groups in G.

Corollary 3.2. *The following conditions are equivalent:*

1. *f_A is a normal soft int-group,*
2. *f_A is soft Abelian,*
3. *f_A is constant in the conjugate class of A, that is, $f_A(xyx^{-1}) = f_A(y)$ for all x,$y \in A$,*
4. *$f_A(xyx^{-1}) \supseteq f_A(y)$ for all x,$y \in A$,*
5. *$f_A(xyx^{-1}) \subseteq f_A(y)$ for all x,$y \in A$.*

Proof. See [10, Teorem 8]. □

Corollary 3.3. *If G is an Abelian group, then any soft int-group in G is a normal soft int-group.*

Corollary 3.4. *If $f_A \in S_G(U)$ and $A \lhd G$, then $f_A \in NS_G(U)$.*

Corollary 3.5. *Let $f_G \in S_G(U)$. Then, $f_{\widetilde{G}}$ and $f_{\left(e_{f_G}\right)(f_G(e))}$ are normal soft int-groups.*

Corollary 3.6. *Let $A_i \leq G$ for all $i \in I$ and $\{f_{A_i} : i \in I\}$ be a family of normal soft int-groups in G. Then, $\widetilde{\bigcap\limits_{i\in I}} f_{A_i} \in NS_G(U)$.*

Theorem 3.7. *Let $A \subseteq G$ and $f_A \in S_G(U)$. Then, f_A is normal soft int-group (Abelian) if and only if $f_A\left([x, y]\right) = f_A\left(e\right)$ for all x,$y \in G$, where $[x, y] = x^{-1}y^{-1}xy$ is commutator of x and y.*

Proof. Let $f_A\left([x, y]\right) = f_A\left(e\right)$ for all x,$y \in G$. Then,

$$\begin{aligned} f_A\left(xy\right) &= f_A\left(\left(yx\right)\left(yx\right)^{-1}\left(xy\right)\right) \\ &\supseteq f_A\left(yx\right) \cap f_A\left(x^{-1}y^{-1}xy\right) \\ &= f_A\left(yx\right) \cap f_A\left(e\right) \\ &= f_A\left(yx\right) \end{aligned}$$

and

$$\begin{aligned} f_A\left(yx\right) &= f_A\left(\left(xy\right)\left(xy\right)^{-1}\left(yx\right)\right) \\ &\supseteq f_A\left(xy\right) \cap f_A\left(y^{-1}x^{-1}yx\right) \\ &= f_A\left(xy\right) \cap f_A\left(e\right) \\ &= f_A\left(xy\right) \end{aligned}$$

so $f_A\left(xy\right) = f_A\left(yx\right)$.

The converse is obvious. □

Theorem 3.8. *[16] Let $N \lhd G$. Then, G/N is Abelian if and only if $G' \subseteq N$, where G' is the commutator subgroup of the group G.*

Theorem 3.9. *Let $f_A \in NS_G(U)$. Then, the quotient group G/e_{f_A} is normal.*

Proof. Assume that $f_A \in NS_G(U)$. Then we have $e_{f_A} \lhd G$ by Corollary 3.5 and e_{f_A} contains commutator subgroup G' of G. So G/e_{f_A} is Abelian by Theorem 3.8, so it is normal. $\square$

If H and K are two subgroups of a group G, then the set $[H, K]$ defined as $[H, K] = \{[h, k] : h \in H \text{ and } k \in K\}$ is a subgroup of G. In addition, we know from group theory that $H \lhd G$ if and only if $[H, G] \leq H$ [14, p.169]. Analogues to this fact we write the following theorem.

Theorem 3.10. *Let $A \subseteq G$. Then, a soft int-group f_A is a normal soft int-group if and only if*

$$f_A([x, y]) \supseteq f_A(x) \tag{1}$$

for all $x, y \in G$.

Proof. Assume that $f_A \in NS_G(U)$. Let x,$y \in G$ then

$$\begin{aligned} f_A([x, y]) &= f_A\left(x^{-1}y^{-1}xy\right) \\ &\supseteq f_A\left(x^{-1}\right) \cap f_A\left(y^{-1}xy\right) \\ &= f_A(x) \cap f_A(x) \\ &= f_A(x). \end{aligned}$$

Conversely, suppose that f_A satisfies (1). Then for all x,$y \in G$, we have

$$\begin{aligned} f_A\left(x^{-1}yx\right) &= f_A\left(yy^{-1}x^{-1}yx\right) \\ &\supseteq f_A(y) \cap f_A([y, x]) \\ &= f_A(y) \end{aligned}$$

by assumption, so f_A is a normal soft int-group by Corollary 3.2 $\square$

Now, we give some properties of normal soft int-groups including α-inclusion.

Let $f_G^{\alpha_i}$ be level subgroups of G, where $\alpha_i \in \mathrm{Im}(f_G)$, for any $i \in I$ and $\alpha_0 \supseteq \alpha_1 \supseteq \cdots \supseteq \alpha_r$. We know that for any α_i, α_j in $\mathrm{Im}(f_G)$, $\alpha_i \subseteq \alpha_j$ implies $f_G^{\alpha_i} \supseteq f_G^{\alpha_j}$ by Corollary 2.16. So any soft int-group on a finite group G gives a chain with subgroups of G as;

$$e_{f_G} = f_G^{\alpha_0} \subseteq f_G^{\alpha_1} \subseteq \cdots \subseteq f_G^{\alpha_r} = G. \tag{2}$$

We denote this chain of level subgroups by $L_f(G)$. It is clear that, all subgroups of a group G need not form a chain. It follows that not all subgroups are level subgroups of a soft int-group.

Definition 3.11. Let $f_G \in S_G(U)$. Then, f_G is called soft level normal subgroup if and only if the number of the level subgroups are finite and $f_G^{\alpha_i} \lhd f_G^{\alpha_{i+1}}$ for each level subgroups in $L_f(G)$.

Theorem 3.12. *Let G be a finite group. Then, any normal soft int-group in G is a soft level normal subgroup of G.*

Proof. Assume that $f_G \in NS_G(U)$ and let $f_G^{\alpha_i}$ be as in (2). So, for any $x \in f_G^{\alpha_i}$,

$$f_G\left(y^{-1}xy\right) = f_G\left(x\right) \tag{3}$$

for all $y \in G$. Hence (3) is true for all $y \in f_G^{\alpha_{i+1}}$, as well. Consequently $f_G^{\alpha_i} \lhd f_G^{\alpha_{i+1}}$. □

Example 3.13. *Every soft int-group of an Abelian finite group is a soft level normal subgroup.*

Example 3.14. *A soft int-group of a finite group may not be a soft level normal subgroup.*

Let $A = \{e, v\} \subset G = D_3$, where $D_3 = \{e, u, u^2, v, vu, vu^2\}$ is the Dihedral group such that $u^3 = v^2 = e$ and $uv = vu^2$. It is clear that $\{e\} \leq A \leq D_3$ is a chain of subgroups. It is easy to see that $f_{\widetilde{A}}$ is a soft int-group in D_3. However, since

$$\phi = f_{\widetilde{A}}(vu) = f_{\widetilde{A}}(vu\left(uu^{-1}\right)) = f_{\widetilde{A}}(vu^2u^{-1}) = f_{\widetilde{A}}(uvu^{-1}) \neq f_{\widetilde{A}}(v) = U,$$

$f_{\widetilde{A}}$ is not a normal soft int-group, so it is not a soft level normal subgroup of G.

But if we choose the subgroup chain $\{e\} \leq \left\{e, u, u^2\right\} \leq D_3$, we observe that all element of chain are soft level normal subgroup.

Theorem 3.15. *Let $f_G \in S(U)$ and $\alpha \in P(U)$. Then, $f_G \in NS_G(U)$ if and only if f_G^{α} is a normal subgroup of G, whenever it is nonempty.*

Proof. By Theorem 2.18 we know that if $f_G \in S_G(U)$ then $f_G^{\alpha} \leq G$ for any nonempty f_G^{α}. Now we assume that $f_G \in NS_G(U)$. Let $x \in G$, then for any $y \in f_G^{\alpha}$ we have $f_G(xyx^{-1}) = f_G(y) \supseteq \alpha$.

Thus, $xyx^{-1} \in f_G^{\alpha}$ and hence $f_G^{\alpha} \lhd G$.

Conversely, by Theorem 2.18 we know that if $f_G^{\alpha} \leq G$, for any nonempty f_G^{α}, then $f_G \in S_G(U)$. Now, suppose $f_G^{\alpha} \lhd G$. Let x,$y \in G$ and $\alpha = f_G(y)$. Then $y \in f_G^{\alpha}$ and so $xyx^{-1} \in f_G^{\alpha}$, since f_G^{α} is normal subgroup of G.

Hence $f_G(xyx^{-1}) \supseteq \alpha = f_G(y)$ and so $f_G \in NS_G(U)$ by Corollary 3.2. □

Corollary 3.16. *Let $f_G \in NS_G(U)$. Then, e-set (e_{f_G}) and support of f_G (f_G^*) are normal subgroups of G.*

Proof. Direct by Theorem 3.15. □

In group theory, a Dedekind group is a group G such that every subgroup of G is normal. All Abelian groups are Dedekind groups. A non-Abelian Dedekind group is called a Hamiltonian group.

Example 3.17. *The quaternion group defined as;*

$$Q = \Big\{-1, i, j, k : (-1)^2 = 1,\ i^2 = j^2 = k^2 = ijk = -1\Big\}$$

is Hamiltonian, where 1 is the identity element and -1 commutes with the other elements of the group.

Theorem 3.18. *G is a Dedekind group if and only if every soft int-group in G is a normal soft int-group.*

Proof. Let G be a Dedekind group and $f_A \in S_G(U)$. We need to show $f_A(y) = f_A(xyx^{-1})$ for all $y \in A$ and $x \in G$. Let $f_A(y) = \alpha_1$ and $f_A(xyx^{-1}) = \alpha_2$. Then $f_A^{\alpha_1}$ and $f_A^{\alpha_2}$ are subgroups of G by Theorem 2.18 and so are normal subgroups of G, since G is a Dedekind group. If $y \in f_A^{\alpha_1}$ then $xyx^{-1} \in f_A^{\alpha_1}$ so $\alpha_2 = f_A(xyx^{-1}) \supseteq \alpha_1$. On the other hand $xyx^{-1} \in f_A^{\alpha_2}$ and since $f_A^{\alpha_2}$ is normal, $y \in f_A^{\alpha_2}$ so $\alpha_1 = f_A(y) \supseteq \alpha_2$. Thus $\alpha_1 = \alpha_2$.

Conversely, suppose that every soft int-group in G is a normal soft int-group. Then, by Theorem 3.15 any subgroup A of a group G can be regarded as a level subgroup of some soft int-group f_A in G. Since every soft int-group in G is a normal soft int-group, then A is normal subgroup of G. Thus G is a Dedekind group. □

Lemma 3.19. *Let A,B $\subseteq G$. If $f_A \in NS_G(U)$, then for any soft set f_B in G, $f_A * f_B = f_B * f_A$.*

Proof. For all $x \in G$, we have

$$(f_A * f_B)(x) = \bigcup \{f_A(u) \cap f_B(v) : uv = x,\ u, v \in G\}$$

and since f_A is normal soft int-group and $uv = x$ implies $u = xv^{-1}$, then

$$\begin{aligned}(f_A * f_B)(x) &= \bigcup \big\{f_A(xv^{-1}) \cap f_B(v) : (xv^{-1})v = x,\ v \in G\big\} \\ &= \bigcup \big\{f_B(v) \cap f_A(v^{-1}x) : v(v^{-1}x) = x,\ v \in G\big\} \\ &= (f_B * f_A)(x).\end{aligned}$$

□

Theorem 3.20. *If $f_A \in NS_G(U)$ and $f_B \in S_G(U)$, then $(f_A * f_B) \in S_G(U)$.*

Proof. Firstly,

$$\begin{aligned}(f_A * f_B) * (f_A * f_B) &= f_A * (f_B * f_A) * f_B \\ &= f_A * (f_A * f_B) * f_B \text{ (by Lemma 3.19)} \\ &= (f_A * f_A) * (f_B * f_B) \\ &\subseteq f_A * f_B \qquad \text{(by Theorem 2.24)}\end{aligned}$$

Secondly, for all $x \in G$

$$\begin{aligned}
(f_A * f_B)\left(x^{-1}\right) &= \bigcup\left\{f_A(u) \cap f_B(v) : uv = x^{-1},\ u, v \in G\right\} \\
&= \bigcup\left\{f_A\left((u^{-1})^{-1}\right) \cap f_B\left((v^{-1})^{-1}\right) : v^{-1}u^{-1} = x,\ u, v \in G\right\} \\
&= \bigcup\left\{f_B(v^{-1}) \cap f_A(u^{-1}) : v^{-1}u^{-1} = x,\ u, v \in G\right\} \\
&= (f_B * f_A)(x) \\
&= (f_A * f_B)(x) \ \text{(by Lemma 3.19).}
\end{aligned}$$

Hence $(f_A * f_B) \in S_G(U)$ by Theorem 2.24. □

Corollary 3.21. *If f_A,$f_B \in NS_G(U)$, then $(f_A * f_B) \in NS_G(U)$.*

Proof. $(f_A * f_B) \in S_G(U)$ by Theorem 3.20, so we should verify that $f_A * f_B$ is a normal soft int-group. For any $x \in G$, we have

$$\begin{aligned}
(f_A * f_B)(x) &= \bigcup\left\{f_A(u) \cap f_B(v) : uv = x,\ u, v \in G\right\} \\
&= \bigcup\left\{f_A(w^{-1}uw) \cap f_B(w^{-1}vw) : \left(w^{-1}uw\right)\left(w^{-1}vw\right) = w^{-1}xw,\ u, v \in G\right\} \\
&= (f_A * f_B)\left(w^{-1}xw\right)
\end{aligned}$$

for all w,$x \in G$. Hence $(f_A * f_B) \in NS_G(U)$. □

Remark 3.22. $(NS_G(U), *)$ is a commutative idempotent semigroup, because

1. $NS_G(U)$ is closed under operation " $*$ ", (by Corollary 3.21)
2. $(NS_G(U), *)$ is commutative, (by Theorem 2.25)
3. $(NS_G(U), *)$ is associative, (by Theorem 2.22)
4. $(NS_G(U), *)$ is idempotent, i.e., $f_G * f_G = f_G$.

Definition 3.23. Let f_A,$f_B \in S_G(U)$. Then, f_A and f_B are called conjugate soft int-groups (with respect to u), if there exists $u \in G$ such that, $f_A(x) = f_B(uxu^{-1})$, for all $x \in G$ and we denote $f_A = f_{B^u}$, where $f_{B^u}(x) = f_B(uxu^{-1})$, for all $x \in G$.

Theorem 3.24. *A soft int-group f_A is a normal soft int-group in G if and only if f_A is constant on each conjugate class of G, that is, $f_{A^u} = f_A$ for all $u \in G$.*

Proof. Suppose $f_A \in NS_G(U)$. Then,

$$f_A\left(uxu^{-1}\right) = f_A(xuu^{-1}) = f_A(x), \text{ for all } x, u \in G.$$

Conversely suppose that f_A is constant on each conjugate class of G. Then

$$f_A(xu) = f_A(xuxx^{-1}) = f_A(x\,(ux)\,x^{-1}) = f_A(ux)$$

for all x,$u \in G$, so $f_A \in NS_G(U)$. □

Definition 3.25. If $f_G \in S_G(U)$, then the set defined as

$$N(f_G) = \{x \in G : f_G(xy) = f_G(yx)\}$$

for all $y \in G$, is called normalizer of f_G in G.

Clearly, for all $f_G \in S_G(U)$, the unit element of a group G is in $N(f_G)$ and if G is Abelian then $N(f_G) = G$.

Corollary 3.26. *If $x \in N(f_G)$, then $x^{-1} \in N(f_G)$.*

Proof. Let $x \in N(f_G)$. Then, for all $u \in G$,

$$f_G(x^{-1}u) = f_G(x^{-1}u\left(xx^{-1}\right)) = f_G(x^{-1}\left(ux\right)x^{-1}) = f_G(x^{-1}\left(xu\right)x^{-1}) = f_G(ux^{-1})$$

so $x^{-1} \in N(f_G)$. $\square$

Corollary 3.27. *If $f_G \in S_G(U)$, then*

$$N(f_G) = \{u \in G : f_{G^u} = f_G\}. \tag{4}$$

Theorem 3.28. *Let $f_G \in S_G(U)$. Then,*

1. *$N(f_G) \leq G$,*
2. *The restriction of f_G to $N\left(f_G\right)$ is a normal soft int-group, that is $f_G|_{N(f_G)} \in NS_G(U)$,*
3. *$f_G \in NS_G(U)$ if and only if $N(f_G) = G$.*

Proof. Let $f_G \in S_G(U)$.

1. Obviously $N(f_G) \neq \phi$, since $e \in N(f_G)$. Let $x, y \in N(f_G)$. Then, for all $u \in G$,

$$\begin{aligned} f_G(\left(xy^{-1}\right)u) &= f_G(x\left(y^{-1}u\right)) \\ &= f_G(x\left(uy^{-1}\right)) \text{ (since } y^{-1} \in N(f_G)) \\ &= f_G((xu)\,y^{-1}) \\ &= f_G((ux)\,y^{-1}) \\ &= f_G(u\left(xy^{-1}\right)) \end{aligned}$$

so $xy^{-1} \in N(f_G)$. Hence $N(f_G)$ is a subgroup of G.

2. $f_G|_{N(f_G)}$ is a soft int-group by Theorem 2.13. Since $N(f_G)$ is Abelian, $f_G|_{N(f_G)}(xy) = f_G|_{N(f_G)}(yx)$, for all x,$y \in N(f_G)$. Hence, $f_G|_{N(f_G)}$ is a normal soft int-group.

3. Suppose that f_G is a normal soft int-group and $u \in G$. Then for any $g \in G$ we have

$$\begin{aligned} f_{G^u}(g) &= f_G(u^{-1}gu) \\ &= f_G\left(u\left(u^{-1}g\right)\right) \text{ (by assumption)} \\ &= f_G\left(g\right). \end{aligned}$$

So $f_{G^u} = f_G$ and hence $u \in N(f_G)$ by (4), which implies $G \subseteq N(f_G)$. Since $N(f_G) \leq G$ then $N(f_G) \subseteq G$ and so $N(f_G) = G$.

Conversely, let $N(f_G) = G$ and $x,y \in G$. Then we have

$$f_G\left(xy\right) = f_G\left(xyxx^{-1}\right) = f_G\left(\left(x^{-1}\right)^{-1}\left(yx\right)x^{-1}\right) = f_{G^{x^{-1}}}(yx). \tag{5}$$

On the other hand, since $N(f_G) = G$, for any $x \in G = N(f_G)$ we have $x^{-1} \in N(f_G)$, and so

$$f_{G^{x^{-1}}}(yx) = f_G\left(yx\right) \tag{6}$$

by the definition of normalizer. Thus $f_G\left(xy\right) = f_G\left(yx\right)$, for all $x,y \in G$ from (5) and (6).

Hence, f_G is a normal soft int-group. $\square$

Theorem 3.29. *Let G be a finite group and $f_G \in S_G(U)$ such that $f_G^* \neq \phi$. Then, the number of distinct conjugate classes of f_G is equal to the index of $N(f_G)$ in G.*

Proof. Since $N(f_G) \leq G$, G can be written as a union of cosets of $N(f_G)$, as

$$G = x_1N(f_G) \cup x_2N(f_G) \cup \ldots \cup x_kN(f_G),$$

where k is the number of distinct cosets, that is $k = \left|G : N(f_G)\right|$. Let $x \in N(f_G)$ and choose i such that $1 \leq i \leq k$. Then for any $g \in G$,

$$\begin{aligned} f_{G^{x_ix}}\left(g\right) &= f_G\left(\left(x_ix\right)^{-1}g\left(x_ix\right)\right) \\ &= f_G\left(x^{-1}\left(x_i^{-1}gx_i\right)x\right) \\ &= f_{G^x}\left(x_i^{-1}gx_i\right) \\ &= f_G\left(x_i^{-1}gx_i\right) \ \text{ (since } x \in N(f_G)) \\ &= f_{G^{x_i}}\left(g\right). \end{aligned}$$

Thus, we have $f_{G^{x_ix}}\left(g\right) = f_{G^{x_i}}\left(g\right)$, for all $x \in N(f_G)$ and $1 \leq i \leq k$.

So any two elements in G, which lie in the same coset $x_iN(f_G)$ give rise to the same conjugate $f_{G^{x_i}}$ of f_G. Now we show that two distinct cosets give two distinct conjugates of f_G. Suppose that $f_{G^{x_i}} = f_{G^{x_j}}$, where $i \neq j$ and $1 \leq i,j \leq k$. Thus, for all $g \in G$,

$$\begin{aligned} f_{G^{x_i}} &= f_{G^{x_j}} \\ &\Leftrightarrow f_{G^{x_i}}\left(g\right) = f_{G^{x_j}}\left(g\right) \\ &\Leftrightarrow f_G\left(x_i^{-1}gx_i\right) = f_G\left(x_j^{-1}gx_j\right). \end{aligned}$$

If we choose $g = x_j t x_j^{-1}$, it follows that

$$f_G\left(x_i^{-1}\left(x_j t x_j^{-1}\right)x_i\right) = f_G\left(x_j^{-1}\left(x_j t x_j^{-1}\right)x_j\right)$$

$$\begin{aligned}
&\Rightarrow \quad f_G\left(\left(x_j^{-1}x_i\right)^{-1} t\left(x_j^{-1}x_i\right)\right) = f_G(t) \text{ for all } t \in G \\
&\Rightarrow \quad f_{G^{x_j^{-1}x_i}}(t) = f_G(t) \text{ for all } t \in G \\
&\Rightarrow \quad x_j^{-1}x_i \in N(f_G) \\
&\Rightarrow \quad x_i N(f_G) = x_j N(f_G).
\end{aligned}$$

However, if $i \neq j$, this is not possible when we consider the decomposition of G as a union of cosets of $N(f_G)$. Hence the number of distinct conjugates of f_G is equal to $|G : N(f_G)|$. □

Theorem 3.30. *Let $f_A \in S_G(U)$ and f_{A^u} be as in Definition (3.23). Then,*

1. $\widetilde{\bigcap}_{u\in G} f_{A^u}$ *is a normal soft int-group,*
2. $\widetilde{\bigcap}_{u\in G} f_{A^u}$ *is the largest normal soft int-group in G, contained in f_A.*

Proof. Let $f_A \in S_G(U)$. Then,

1. $\widetilde{\bigcap}_{u\in G} f_{A^u}$ is a soft int-group, since f_{A^u} are soft int-groups, for all $u \in G$, by Theorem 2.14. Now, for all x,$y \in G$

$$\begin{aligned}
\bigcap_{u\in G} f_{A^u}\left(xyx^{-1}\right) &= \bigcap_{u\in G} f_A\left(u\left(xyx^{-1}\right)u^{-1}\right) \\
&= \bigcap_{u\in G} f_A\left((ux)\,y\,(ux)^{-1}\right) \\
&= \bigcap_{u\in G} f_{A^{ux}}(y) \\
&= \bigcap_{u\in G} f_{A^u}(y)\ .
\end{aligned}$$

   since f_{A^u} and $f_{A^{ux}}$ are in the same conjugate class of f_A, for all $x \in G$. Thus, $\widetilde{\bigcap}_{u\in G} f_{A^u}$ is a normal soft int-group, by Theorem 3.24.

2. Let f_B be a normal soft int-group satisfying $f_B \subseteq f_A$. Then $f_B = f_{B^u} \widetilde{\subseteq} f_{A^u}$ for all $u \in G$, by assumption. Thus, $f_B \subseteq \widetilde{\bigcap}_{u\in G} f_{A^u}$. Therefore $\widetilde{\bigcap}_{u\in G} f_{A^u}$ is the largest normal soft int-group in G, contained in f_A.

□

Now, we introduce the notion of coset.

Let $f_G \in S_G(U)$ and $a \in G$. Then the soft subsets $f_{a(f_G(e))} * f_G$ and $f_G * f_{a(f_G(e))}$ are referred to as the left coset and right coset of f_G with respect to a.

From Corollary 2.23, we have that $\left(f_{a(f_G(e))} * f_G\right)(y) = f_G\left(a^{-1}y\right)$ and $\left(f_G * f_{a(f_G(e))}\right)(y) = f_G\left(ya^{-1}\right)$, since $f_G(e) \supseteq f_G(x)$, for all $x \in G$. On the base of these facts, the following definition is given:

Definition 3.31. [10] Let $f_G \in S_G(U)$ and $a \in G$. Then, soft left coset of f_G, denoted by af_G, is defined by the approximation function $(af_G)(x) = f_G(a^{-1}x)$ for all $x \in G$.

Similarly, right coset of f_G can be defined by the approximation function $(f_Ga)(x) = f_G(xa^{-1})$ for all $x \in G$ and denoted by f_Ga.

If $f_G \in NS_G(U)$, then soft the left coset is equal to the soft right coset. Thus in this case, we call only soft coset and denote by af_G.

Definition of coset above is analogues to definition of classical algebra as follows:

Let $H \leq G$ and f_H be the characteristic function of H, that is

$$f_H(x) = \begin{cases} U & for\ x \in H \\ \phi & for\ x \in G \backslash H \end{cases}.$$

It is well known that for any $a \in G$, $aG = G$.

Now if $g \in H$, then $ag \in aH$, so

$$af_H(ag) = f_H\left(a^{-1}ag\right) = f_H\left(g\right) = U.$$

If $g \notin H$, then $ag \notin aH$, and so

$$af_H(ag) = f_H\left(a^{-1}ag\right) = f_H\left(g\right) = \phi.$$

Thus, it follows that af_H is a function on G, such that

$$af_H(x) = \begin{cases} U & for\ x \in aH \\ \phi & for\ x \in G \backslash \left(aH\right) \end{cases}.$$

This shows that af_H is the characteristic function of aH.

Proposition 3.32. *Let $f_G \in S_G(U)$. Then, there is a one-to-one correspondence between the set of right cosets and the set of left cosets of f_G in G.*

Theorem 3.33. *Let $f_G \in NS_G(U)$.Then, for any $a \in G$*

$$af_G(ga) = af_G\left(ag\right) = f_G\left(g\right), \text{ for all } g \in G.$$

Proof. Let $f_G \in NS_G(U)$. Then, for any $a \in G$

$$af_G(ga) = f_G\left(a^{-1}ga\right) = f_G\left(g\right)$$

since $f_G \in NS_G(U)$. The other part is similar. □

Theorem 3.34. *[10] Let $f_G \in S_G(U)$. Then, $af_G = bf_G \Leftrightarrow ae_{f_G} = be_{f_G}$ for all $a,b \in G$.*

Theorem 3.35. *[10] Let $f_G \in NS_G(U)$. If $af_G = bf_G$, then $f_G(a) = f_G(b)$ for any $a,b \in G$.*

Now, we introduce the notion of quotient groups.

Theorem 3.36. *Let $f_G \in NS_G(U)$ and define a set $G/f_G = \{xf_G : x \in G\}$. Then, the following assertions hold:*

1. *$(xf_G) * (yf_G) = (xyf_G)$, for all $x,y \in G$,*
2. *$(G/f_G, *)$ is a group. Moreover, if G is Abelian then so is G/f_G.*

Proof. Let $f_G \in NS_G(U)$.

1. For all $x,y \in G$,

$$\begin{aligned}
(xf_G) * (yf_G) &= \left(f_{x(f_G(e))} * f_G\right) * \left(f_{y(f_G(e))} * f_G\right) \\
&= f_{x(f_G(e))} * \left(f_G * f_{y(f_G(e))}\right) * f_G \\
&= f_{x(f_G(e))} * \left(f_{y(f_G(e))} * f_G\right) * f_G \\
&= \left(f_{x(f_G(e))} * f_{y(f_G(e))}\right) * (f_G * f_G) \\
&= \left(f_{x(f_G(e))} * f_{y(f_G(e))}\right) * f_G \text{ (by Remark 3.22)} \\
&= f_{(xy)(f_G(e))} * f_G \text{ (by Theorem 2.21)} \\
&= xyf_G.
\end{aligned}$$

2. $(G/f_G, *)$ is closed under the operation "$*$" by part 1 and it is associative by Theorem 2.22. Now, for all $x \in G$

$$f_G * xf_G = ef_G * xf_G = (ex)\, f_G = xf_G = (xe)\, f_G = xf_G * ef_G = xf_G * f_G$$

so, $f_G = ef_G$ is an identity element of G/f_G. In addition, for all $x \in G$

$$\left(x^{-1}f_G\right) * (xf_G) = \left(x^{-1}x\right) f_G = ef_G = \left(xx^{-1}\right) f_G = (xf_G) * \left(x^{-1}f_G\right)$$

so, $\left(x^{-1}f_G\right)$ is the inverse of (xf_G). Hence $(G/f_G, *)$ is a group.

Moreover if G is Abelian then for all $x,y \in G$

$$xf_G * yf_G = xyf_G = yxf_G = yf_G * xf_G$$

so G/f_G is Abelian. ∎

Definition 3.37. Let $f_G \in NS_G(U)$. Then, the group G/f_G defined in Theorem 3.36 is called the quotient (or factor) group of G relative to the normal soft int-group f_G.

Theorem 3.38. *Let $f_G \in NS_G(U)$. Then, $G/f_G \cong G/e_{f_G}$.*

Proof. Since $f_G \in NS_G(U)$, e_{f_G} is a normal subgroup of G, by Corollary 3.16 and hence G/e_{f_G} is a quotient group. In addition, G/f_G is a group by Theorem 3.36. Now, define a map $\varphi : G/f_G \to G/e_{f_G}$ by setting $\varphi(xf_G) = xe_{f_G}$. Firstly, φ is a homomorphism since, for all xf_G,$yf_G \in G/f_G$

$$\varphi\left((xf_G) * (yf_G)\right) = \varphi\left(xyf_G\right) = (xy)\, e_{f_G} = (xe_{f_G})\,(ye_{f_G}) = \varphi\left(xf_G\right)\varphi\left(yf_G\right).$$

On the other hand, φ is a bijection by Theorem 3.34.

Hence $G/f_G \cong G/e_{f_G}$. $\square$

Theorem 3.39. *Let $f_G \in NS_G(U)$ and define a soft set $f_G^{(*)}$ on G/f_G by, $f_G^{(*)}(xf_G) = f_G(x)$, for all $x \in G$. Then, $f_G^{(*)}$ is a normal soft int-group in G/f_G.*

Proof. Firstly, $f_G^{(*)}$ is well defined since, for any xf_G,$yf_G \in G/f_G$

$$\begin{aligned}(xf_G) &= (yf_G) \Rightarrow f_G(x) = f_G(y) \text{ (by the Theorem 3.35)}\\ &\Rightarrow f_G^{(*)}(xf_G) = f_G^{(*)}(yf_G).\end{aligned}$$

Secondly, for all x,$y \in G$

$$\begin{aligned}f_G^{(*)}\left((xf_G) * (yf_G)\right) &= f_G^{(*)}(xyf_G)\\ &= f_G(xy)\\ &\supseteq f_G(x) \cap f_G(y)\\ &= f_G^{(*)}(xf_G) \cap f_G^{(*)}(yf_G)\end{aligned}$$

and for all $x \in G$,

$$f_G^{(*)}\left((xf_G)^{-1}\right) = f_G^{(*)}\left(x^{-1}f_G\right) = f_G\left(x^{-1}\right) = f_G(x) = f_G^{(*)}(xf_G)$$

thus $f_G^{(*)} \in S_{G/f_G}(U)$. Thirdly, for all x,$y \in G$,

$$\begin{aligned}f_G^{(*)}\left((xf_G) * (yf_G)\right) &= f_G^{(*)}(xyf_G)\\ &= f_G(xy)\\ &= f_G(yx)\\ &= f_G^{(*)}(yxf_G)\\ &= f_G^{(*)}\left((yf_G) * (xf_G)\right)\end{aligned}$$

hence $f_G^{(*)} \in NS_{G/f_G}(U)$. $\square$

Theorem 3.40. *Let $f_A, f_B \in S_H(U)$ be such that $f_A \widetilde{\subseteq} f_B$, H be a group and $\varphi : G \to H$ be a homomorphism. Then $\varphi^{-1}(f_A) \widetilde{\subseteq} \varphi^{-1}(f_B)$.*

Proof. For all $x \in G$

$$\varphi^{-1}(f_A)(x) = \ f_A(\varphi(x)) \subseteq \ f_B(\varphi(x)) = \varphi^{-1}(f_B)(x).$$

So, $\varphi^{-1}(f_A)\tilde{\subseteq}\varphi^{-1}(f_B)$. □

Theorem 3.41. *Let $f_G \in NS_G(U)$ and H be a group. If φ is an epimorphism from G onto H then, $\varphi(f_G) \in NS_H(U)$.*

Proof. We have $\varphi(f_G) \in S_H(U)$ (see [10] Theorem 19). Since φ is onto there exist u,$v \in G$ such that $\varphi(u) = x$ and $\varphi(v) = y$ for any x,$y \in H$. Thus for all x,$y \in H$,

$$\begin{aligned}
\varphi(f_G)(xy) &= \bigcup\{f_G(w) : w \in G,\ \varphi(w) = xy\} \\
&= \bigcup\{f_G(uv) : uv \in G,\ \varphi(uv) = xy\} \\
&= \bigcup\{f_G(uv) : vu \in G,\ \varphi(u)\ \varphi(v) = xy\} \ \text{(since } \varphi \text{ is a homomorphism)} \\
&= \bigcup\{f_G(uv) : vu \in G,\ \varphi(u) = x \text{ and } \varphi(v) = y\} \\
&= \bigcup\{f_G(vu) : vu \in G,\ \varphi(v)\varphi(u) = yx\} \ \text{(since } f_G \in NS_G(U)) \\
&= \bigcup\{f_G(vu) : vu \in G,\ \varphi(vu) = yx\} \\
&= \varphi(f_G)(yx).
\end{aligned}$$

Hence $\varphi(f_G)$ is a normal soft int-group. □

Theorem 3.42. *Let H be a group and $f_H \in NS_H(U)$. If φ is a homomorphism from G into H, then $\varphi^{-1}(f_H) \in NS_G(U)$.*

Proof. We have $\varphi^{-1}(f_H) \in S_G(U)$ (see [10] Theorem 20). For all x,$y \in G$

$$\begin{aligned}
\varphi^{-1}(f_H)(xy) &= f_H(\varphi(xy)) \\
&= f_H(\varphi(x)\varphi(y)) \\
&= f_H(\varphi(y)\varphi(x)) \ \text{(since } f_H \in NS_H(U)) \\
&= f_H(\varphi(yx)) \\
&= \varphi^{-1}(f_H)(yx).
\end{aligned}$$

Hence $\varphi^{-1}(f_H)$ is normal soft int-group. □

Lemma 3.43. *Let $\varphi : A \to B$ be a function. Then, for all $f_B \in S(U)$, $f_B\tilde{\supseteq}\varphi\left(\varphi^{-1}(f_B)\right)$.*

In particular, if φ is a surjective function, then $f_B = \varphi\left(\varphi^{-1}(f_B)\right)$.

Theorem 3.44. *Let $f_B \in NS_H(U)$, H be a group and $\varphi : G \to H$ be a homomorphism. Then, $\varphi\left(\varphi^{-1}(f_B)\right) \in NS_H(U)$.*

Proof. Let $f_B \in NS_H(U)$. Then,

$$\begin{aligned}
\varphi\left(\varphi^{-1}(f_B)\right)\left(xyx^{-1}\right) &= \cup\left\{\varphi^{-1}(f_B)(z) : z \in G,\ \varphi\left(z\right) = xyx^{-1}\right\} \\
&= \cup\left\{f_B\left(\varphi(z)\right) : z \in G,\ \varphi\left(z\right) = xyx^{-1}\right\} \\
&= \cup\left\{f_B\left(\varphi(xyx^{-1})\right)\right\} \\
&= \cup\left\{f_B\left(\varphi(x)\varphi(y)\varphi(x)^{-1}\right)\right\} \\
&\supseteq \cup\left\{f_B\left(\varphi(y)\right)\right\} \quad (\text{since } f_B \in NS_H(U)) \\
&= \cup\left\{\varphi^{-1}(f_B)(y)\right\} \\
&= \varphi\left(\varphi^{-1}(f_B)\right)(y) \quad (\text{by Definition 2.12})
\end{aligned}$$

for all $x, y \in G$. □

4. Conclusion

In this paper, we studied on normal soft int-groups and investigate relations with α-inclusion and soft product. Then, we define normalizer, quotient group and give some theorems concerning these concepts. For future works, it is possible to study on isomorphism theorems and other concepts of group theory.

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